Time-dependent equation for the intensity in the diffusion limit using a higher-order angular expansion

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(Received 13 August 1998; revised manuscript received 8 January 1999)

The first two terms in the spherical-harmonic expansion (the P_1 approximation) of the radiative transfer equation yield the diffusion equation. This approximation applies to multiple scattering and results in a solution for the energy density, the gradient of which is proportional to the light intensity. In this work a higher-order spherical-harmonic expansion of the radiative transfer equation is developed. This equation applies to the radiant intensity rather than the energy density. The equation can be decomposed into two terms: a propagator term obtained from the determinant of the coupled equations describing the individual components of the intensity, and a mixing matrix that describes the cross coupling between different orders of the expansion. Using the Fourier transform, an approximation based on expanding at small wave vectors k leads to an equation similar to the diffusion equation. The equation is expected to predict the intensity for multiple scattering at earlier times and shorter distances than the diffusion equation can. The notion of an equivalent wave field is introduced. [S1063-651X(99)08506-2]

PACS number(s): 05.60.−k, 02.90.+p

I. INTRODUCTION

The radiative transfer equation (RTE) and the diffusion equation (DE) to which it reduces in the case of many scatterings provide the basis for describing many physical phenomena. (In this paper I consider the phenomenon of unpolarized light transport.) Analytical solutions of the RTE exist only for restricted cases, and one usually has to resort to numerical methods. One difficulty of the problem is the dimensionality of the system. Besides the position vector and time, each point in space has an associated vector describing the direction of propagation of light at that point. Thus, while the diffusion equation has three spatial dimensions and one of time, the RTE has two additional dimensions.

The problem is usually to investigate the angular distribution using some limiting approximation. The most common approximations are the uniform intensity approximation, leading to the diffusion equation, and the forward-backward dominant scattering approximation, leading to the two-stream model. The DE follows from the RTE under the assumption that after a sufficiently large number of collisions the intensity is almost isotropic and thus provides an asymptotic approximation applicable to later times. Large numbers of physical systems obey the DE independent of the details of the RTE. The diffusion approximation fails for intermediate scattering ranges [1] that are appropriate for many practical applications such as shallow medical imaging, longer-range underwater imaging, and electron transport in small semiconductor devices. Second-order corrections in time to the DE appear occasionally in the literature [2–4], but the validity of these corrections has been debated for the past 20 years [5–9]. A leading reason for the confusion in this matter is the lack of a general solution of the RTE.

II. THEORY

In this paper I derive an equation analogous to the DE that includes the angular distribution of the light intensity. Such an equation is expected to be valid at shorter distances and earlier times than the diffusion equation as it does not assume times or distances sufficient to form a uniform intensity distribution. The outline of the derivation is as follows. First the RTE is represented using the spherical-harmonic expansion (SHE) for the angular intensity with coefficients that are time and space dependent. The resulting representation, the determinant of the coefficient matrix, is a higher-order differential equation similar to the DE. An expansion of the inverse matrix using the perturbation method is then used to determine the mixing of different orders of the source. Summation of terms eliminates the SHE, leading to a differential equation of order L independent of the intermediate SHE.

We start from the time-dependent radiative transfer equation for the intensity I at time t, position $\boldsymbol{r}$, in the direction $\boldsymbol{\Omega}$,

$$\left[\frac{1}{c}\frac{\partial}{\partial t}+\boldsymbol{\Omega}\cdot\nabla+\sigma_t(\boldsymbol{r})\right]I(\boldsymbol{r},t,\boldsymbol{\Omega}) - \frac{\sigma_s(\boldsymbol{r})}{4\pi}\int I(\boldsymbol{r},t,\boldsymbol{\Omega}')p(\boldsymbol{r},\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}')d\Omega' = \varepsilon(\boldsymbol{r},t,\boldsymbol{\Omega}), \tag{1}$$

where σ_s and σ_a are the scattering and absorption coefficients, $\sigma_t=\sigma_s+\sigma_a$, and $p(\boldsymbol{r},\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}')$ is the phase function. The first term on the left-hand side of Eq. (1) describes the time evolution of attenuated ballistic photons, the second term is the scattering integral, and the term on the right-hand side is the source term.

The SHE has the form [10]

$$I_l^m(\boldsymbol{r},t)=\int I(\boldsymbol{r},t,\boldsymbol{\Omega})Y_l^{m*}(\boldsymbol{\Omega})d\Omega, \tag{2a}$$

$$I(\boldsymbol{r},t,\boldsymbol{\Omega})=\sum_{l=0}^{\infty}\sum_{m=-l}^{m=l}\frac{2l+1}{4\pi}I_l^m(\boldsymbol{r},t)Y_l^m(\boldsymbol{\Omega}), \tag{2b}$$

with

$$p(\boldsymbol{r},\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}')=\sum_{l=0}^{\infty}\frac{4\pi}{2l+1}p_l(\boldsymbol{r})P_l(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}'). \tag{3}$$

$P_l(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}')$ is the Legendre polynomial of the cosine of the angle between $\boldsymbol{\Omega}$ and $\boldsymbol{\Omega}'$. In this work, in order to apply the Laplace transform, we will assume that the phase function and the scattering and absorption coefficients are taken to be nonlocal, or independent of $\boldsymbol{r}$. Using the addition theorem for spherical harmonics we get

$$p(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}')=\sum_{l=0}^{\infty}\sum_{m=-l}^{m=l}p_lY_l^m(\boldsymbol{\Omega})Y_l^{m*}(\boldsymbol{\Omega}'). \tag{4}$$

Using the identity $\boldsymbol{\Omega}=(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta)$, we obtain from Eq. (1) the equivalent set of coupled differential equations [11]

$$\begin{aligned}&\left[\frac{1}{c}\frac{\partial}{\partial t}+\sigma_t-\sigma_sp_l\right]I_l^m(\boldsymbol{r},t)+\frac{1}{2l+1}\left\{\sqrt{(l-m+1)(l+m+1)}\,\frac{\partial}{\partial z}I_{l+1}^m(\boldsymbol{r},t)+\sqrt{(l-m)(l+m)}\,\frac{\partial}{\partial z}I_{l-1}^m(\boldsymbol{r},t)\right.\\&\quad-\frac{1}{2}\sqrt{(l+m-1)(l+m)}\left[\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right]I_{l-1}^{m-1}(\boldsymbol{r},t)+\frac{1}{2}\sqrt{(l-m+1)(l-m+2)}\left[\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right]I_{l+1}^{m-1}(\boldsymbol{r},t)+\frac{1}{2}\sqrt{(l-m-1)(l-m)}\\&\quad\times\left.\left[\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right]I_{l-1}^{m+1}(\boldsymbol{r},t)-\frac{1}{2}\sqrt{(l+m+1)(l+m+2)}\left[\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right]I_{l+1}^{m+1}(\boldsymbol{r},t)\right\}=\varepsilon_l^m(\boldsymbol{r},t).\end{aligned} \tag{5}$$

The spherical harmonics are eigenvalues of all the terms except the propagation term, which couples adjacent l's and m's. Recognizing this, we expect that any approximation based on spherical harmonics will be a good approximation for scattering-dominated propagation, but will perform poorly in the ballistic regime. This conclusion is consistent with the inefficacy of the SHE in describing transport in the case of a strongly peaked phase function.

Using matrix notation and applying the Fourier transform we write

$$\boldsymbol{D}(\boldsymbol{s},s_t)\hat{\boldsymbol{I}}(\boldsymbol{s},s_t)=\hat{\varepsilon}(\boldsymbol{s},s_t,\boldsymbol{\Omega}), \tag{6a}$$

where $s_t=i\omega$ and $\mathbf{s}=(s_x,s_y,s_z)=i(k_x,k_y,k_z)$, bold characters represent a vector or a matrix, and the caret represents the Fourier (or Laplace) transform; special notation will be used for functions that are defined only in the transform domain. The indices of the matrix elements are combinations of all possible l's and m's. (An effective way of counting l's and m's will be introduced later.) The operator $\boldsymbol{D}(\boldsymbol{s},s_t)$ in Eq. (6a) can be written as

$$\boldsymbol{D}(\boldsymbol{s},s_t)=\boldsymbol{O}(s_t)+\boldsymbol{B}(\boldsymbol{s}), \tag{6b}$$

where the matrix

$$O(s_t)_{(l,m),(l',m')}=\left[\frac{s_t}{c}+\sigma_t-\sigma_l\right]\delta(l,l')\,\delta(m,m') \tag{7}$$

is diagonal and $\boldsymbol{B}$ is a sparse off-diagonal matrix representing $\boldsymbol{\Omega}\cdot\nabla$, which is the spatial ballistic part of the propagation. A formal solution for $\hat{\boldsymbol{I}}$ is given by

$$\hat{\boldsymbol{I}}=\boldsymbol{D}^{-1}\hat{\boldsymbol{\varepsilon}}. \tag{8}$$

We invert the matrix using its determinant and the adjugate matrix. We calculate the determinants directly and we approximate the adjugate matrix by expanding $\boldsymbol{D}$ into an infinite matrix series. Multiplying both sides of Eq. (8) by the determinant of $\boldsymbol{D}$, we obtain a differential equation for $\boldsymbol{I}(\boldsymbol{r},t)$. In calculating the determinant we can take advantage of some properties of Eq. (5). First, the matrix $\boldsymbol{D}$ transforms linearly as $\boldsymbol{D}\rightarrow\boldsymbol{ODO}^T$ under an arbitrary rotation, where $\boldsymbol{O}$ is a linear representation of the orthogonal group. Since the determinant of the matrix $\boldsymbol{O}$ is equal to unity, the determinant of the matrix $\boldsymbol{D}$ is invariant under an arbitrary rotation and hence the only terms that are functions of $\boldsymbol{s}$ have the form $f(\boldsymbol{s}^2)$, where $\boldsymbol{s}^2=s_x^2+s_y^2+s_z^2$; therefore, it is sufficient to calculate the s_z contribution and deduce the rest. Equation (5) calculated in the z direction simplifies to

$$\begin{aligned}&\left[\frac{1}{c}\frac{\partial}{\partial t}+\sigma_t-\sigma_sp_l\right]I_l^m(\boldsymbol{r},t)\\&\quad+\frac{1}{2l+1}\left[\sqrt{(l-m+1)(l+m+1)}\,\frac{\partial}{\partial z}I_{l+1}^m(\boldsymbol{r},t)\right.\\&\quad\left.+\sqrt{(l-m)(l+m)}\frac{\partial}{\partial z}I_{l-1}^m(\boldsymbol{r},t)\right]=\varepsilon_l^m(\boldsymbol{r},t).\end{aligned} \tag{9}$$

Using the Fourier transform, Eq. (9) can be represented as a matrix operation. This matrix can be decomposed into tridiagonal block matrices by grouping elements with the same m, while the l values range from $|m|$ to L, where L is the upper bound of the approximation.

The submatrices corresponding to a given m value have the form

$$\boldsymbol{D}_{m,L}=\boldsymbol{O}_{m,L}+\boldsymbol{B}_{m,L}, \tag{10a}$$

where

$$\boldsymbol{O}_{m,L}=\begin{bmatrix} O_{|m|} & 0 & 0 & & & \\ 0 & O_{|m|+1} & & & & \\ 0 & & & & 0 & 0 \\ & & & O_{L-2} & 0 & 0 \\ & & 0 & 0 & O_{L-1} & 0 \\ & & 0 & 0 & 0 & O_L \end{bmatrix} \tag{10b}$$

and

$$\boldsymbol{B}_{m,L}=\begin{bmatrix} 0 & \frac{1}{\sqrt{2|m|+1}} & 0 & & & \\ \frac{1}{\sqrt{2|m|+1}} & 0 & & & & \\ 0 & & & & 0 & 0 \\ & & & 0 & \frac{\sqrt{(L-1)^2-m^2}}{2L-3} & 0 \\ & & 0 & \frac{\sqrt{(L-1)^2-m^2}}{2L-1} & 0 & \frac{\sqrt{L^2-m^2}}{2L-1} \\ & & 0 & 0 & \frac{\sqrt{L^2-m^2}}{2L+1} & 0 \end{bmatrix} s_z . \tag{10c}$$

From Eq. (10) we find by expanding through the last column and row

$$\det(\boldsymbol{D}_{m,L})=\det(\boldsymbol{D}_{m,L-1})O_L-\frac{L^2-m^2}{4L^2-1}\det(\boldsymbol{D}_{m,L-2})s_z^2, \tag{11}$$

where $\det(\boldsymbol{A})$ is the determinant of $\boldsymbol{A}$.

If we define

$$A_{m,L}=\frac{L^2-m^2}{4L^2-1}\frac{1}{O_L O_{L-1}}, \tag{12}$$

then

$$\det(\boldsymbol{D}_{m,m+n})=\left[\sum_{n=0}^{N} C_{m,n}(-1)^n s_z^{2n}\right]\prod_{l=|m|}^{|m|+n} O_l+1. \tag{13}$$

$C_{m,n}$ is the sum of all products of n $A_{m,l}$ terms with no two l's having adjacent values and $2N\leqslant L-|m|$. The $C_{m,n}$ are functions of s_t. The total determinant up to order L is given by

$$\det(\boldsymbol{D}_L)=\prod_{m=-L}^{L}\det(\boldsymbol{D}_{m,L})[\det(\boldsymbol{D}_{-m,L})=\det(\boldsymbol{D}_{m,L})]. \tag{14}$$

Taking the inverse Laplace transform and recalling that s_z^2 can be replaced by $s_x^2+s_y^2+s_z^2$, we obtain a differential operator in the space-time domain with the space part consisting of the operator $[\nabla^2]^n$. Equation (14) is a generalized diffusion operator of order L that accounts for scattering up to order L (the diffusion equation itself is first order). The terms on the left-hand side of the equation have the form

$$\sum_n f_n(s_t)\nabla^{2n}\tilde{I}(\boldsymbol{r},s_t).$$

The tilde here denotes the partial transform (in t but not in $\boldsymbol{r}$). Using separation of variables, it is possible to show that if we expand $\tilde{I}(\boldsymbol{r},\boldsymbol{\Omega},s_t)$ in spherical harmonics, repeating the operation ∇^2 will introduce derivatives only in $\boldsymbol{r}$, not in $\boldsymbol{\Omega}$. The above property reduces the problem of solving Eq. (14) to one of solving a set of differential equations in $\boldsymbol{r}$.

It is necessary to determine the initial conditions of the above differential equation. This is done by writing Eq. (8) as

$$\det(\boldsymbol{D}_L)\boldsymbol{I}=\boldsymbol{Ad}_L\cdot\hat{\varepsilon}. \tag{15}$$

where $\boldsymbol{Ad}_L$ is the Lth adjugate matrix of $\boldsymbol{D}$, $\boldsymbol{Ad}_L$. The determinant $\det(\boldsymbol{D}_L)$ and the adjugate matrix $\boldsymbol{Ad}_L$ have a major distinction. While the determinant that operates on $\boldsymbol{I}$ causes it to propagate, the adjugate matrix operates on $\hat{\varepsilon}$ and depends only on local values of $\hat{\varepsilon}$. $\det(\boldsymbol{D}_L)$ is the propagator of the equation while the adjugate matrix mixes various orders of the propagating terms. For small values of L, direct calculation of the adjugate matrix $\boldsymbol{Ad}_L$ is possible; for higher orders we introduce an approximation.

To evaluate the right-hand side of Eq. (15) we use a second method of inverting Eq. (8). Using the identity

$$(\boldsymbol{O}+\boldsymbol{B})^{-1}=\sum_{n=0}^{\infty} \boldsymbol{O}^{-1}\cdot(-\boldsymbol{B}\cdot\boldsymbol{O}^{-1})^{n}, \tag{16a}$$

which holds when

$$\lim_{N\to\infty}[(\boldsymbol{B}\cdot\boldsymbol{O}^{-1})^{N}]=\boldsymbol{0} \tag{16b}$$

($\boldsymbol{O}$ is diagonal with a trivial inverse), combining Eqs. (8) and (16), we obtain the formal solution

$$\tilde{I}=\boldsymbol{O}^{-1}\cdot\sum_{n=0}^{\infty}(-\boldsymbol{B}\cdot\boldsymbol{O}^{-1})^{n}\cdot\tilde{\varepsilon}. \tag{17}$$

The operator $\boldsymbol{B}$ was recognized earlier as $\boldsymbol{\Omega}\cdot\nabla$. We must examine the effect of $\boldsymbol{O}^{-1}$ in the space domain. The operator $\boldsymbol{O}^{-1}$ operating on X can be expanded using Eq. (2b) to give

$$\tilde{x}(\boldsymbol{r},s_t,\boldsymbol{\Omega})=\sum_{l=0}^{\infty}\frac{2l+1}{4\pi}\frac{1}{O_l(s_t)}\int\tilde{X}(\boldsymbol{r},s_t,\boldsymbol{\Omega}')\times\sum_{m=-l}^{l}Y_l^{m*}(\boldsymbol{\Omega})Y_l^{m}(\boldsymbol{\Omega}')d\Omega'. \tag{18}$$

From the addition theorem for spherical harmonics

$$P_l(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}')=\frac{4\pi}{2l+1}\sum_{m=-l}^{l}Y_l^{m*}(\boldsymbol{\Omega})Y_l^{m}(\boldsymbol{\Omega}'), \tag{19}$$

where P_l is the Legendre polynomial of degree l, the operator expression $\boldsymbol{O}^{-1}\cdot\tilde{\boldsymbol{X}}$ can be represented in expanded form as

$$\tilde{x}(\boldsymbol{r},s_t,\boldsymbol{\Omega})=\sum_{l=0}^{\infty}\frac{1}{O_l(s_t)}\int\tilde{X}(\boldsymbol{r},s_t,\boldsymbol{\Omega}')P_l(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}')d\Omega'. \tag{20}$$

The quantity

$$Y_l(\boldsymbol{\Omega})=\int Y(\boldsymbol{\Omega}')P_l(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}')d\Omega' \tag{21a}$$

is, by definition, the lth term of the Laplace series expansion [12] of the function Y,

$$Y(\boldsymbol{\Omega})=\sum_{l=0}^{\infty}Y_l(\boldsymbol{\Omega}). \tag{21b}$$

The quantity $Y_l(\boldsymbol{\Omega})$ is called the lth spherical harmonic of the function $Y(\boldsymbol{\Omega})$ [13]. Using Eqs. (17)–(21), we have obtained a formal solution of the radiative transfer equation

$$\tilde{I}(\boldsymbol{r},s_t,\boldsymbol{\Omega})=\sum_{i=0}^{\infty}\sum_{\lambda=0}^{\infty}\frac{1}{O_\lambda(s_t)}\int A_i(\boldsymbol{r},s_t,\boldsymbol{\Omega}')P_\lambda(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}')d\Omega', \tag{22a}$$

$$A_0(\boldsymbol{r},s_t,\boldsymbol{\Omega})=\sum_{\lambda=0}^{\infty}\frac{1}{O_\lambda(s_t)}\int\tilde{\varepsilon}(\boldsymbol{r},s_t,\boldsymbol{\Omega}')P_\lambda(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}')d\Omega', \tag{22b}$$

$$A_{i+1}(\boldsymbol{r},s_t,\boldsymbol{\Omega})=\sum_{\lambda=0}^{\infty}\frac{1}{O_\lambda(s_t)}\times\int(-\boldsymbol{\Omega}\cdot\nabla)A_i(\boldsymbol{r},s_t,\boldsymbol{\Omega}')P_\lambda(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}')d\Omega'. \tag{22c}$$

Equations (22a)–(22c) represent a recursive relation; they appear to be an exact solution, but further examination reveals some problems. In analogy with the adjugate matrix in Eq. (15), Eqs. (22a)–(22c) are composed of local higher-order spatial derivatives of the source; those terms do not propagate away from the source. The utility of Eqs. (22a)–(22c) lies in their ability to yield an approximation for the adjugate matrix that describes the mixing of different orders L of the propagating terms. The propagation of the intensity is described by the determinant $\boldsymbol{Ad}_L$. It is possible to solve first for the effect of the propagator $\boldsymbol{Ad}_L$ on the source and then apply Eq. (17) or (22) to calculate cross terms of the propagation. This method is attractive for cases where all orders l of the source term have the same time and spatial dependence and one solution applies to all orders of the source.

III. DISCUSSION

The first observation we make is that in the transform domain, in all the equations derived from the RTE, including the RTE itself, the transform variable s_t appears in the form $s_t/c+\sigma_t$. Using the properties of the Laplace transform, we note that if $\tilde{X}(s_t)$ is the transform of $X(t)$, then the inverse transform of $\tilde{X}(s_t/c+\sigma_t)$ is $cX(ct)\exp(-\sigma_t ct)$. As a result, the decay of the intensity is preceded by the exponential factor $\exp(-\sigma_t ct)$. This term dominates the decay at early times, but is still present at later times, including when this approximation applies. Using spherical harmonics to represent both the phase function and the scattered intensity has its limitations as both functions tend to be forward peaked while the SHE is good in describing smooth functions; therefore, some modification of the phase function and the RTE can be useful. First we write

$$\sigma_s p(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}')=\sigma_{se}p_e(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}')+\sigma_f\delta(|\boldsymbol{\Omega}-\boldsymbol{\Omega}'|).$$

With proper choice of σ_f, p_e can be approximated using low L values; Eq. (1) then becomes

$$\left[\frac{1}{c}\frac{\partial}{\partial t}+\boldsymbol{\Omega}\cdot\nabla+(\sigma_t-\sigma_f)\right]I(\boldsymbol{r},t,\boldsymbol{\Omega})-\frac{\sigma_{se}}{4\pi}\int I(t,\boldsymbol{\Omega}')p_e(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}')d\Omega'=\varepsilon(\boldsymbol{r},t,\boldsymbol{\Omega}). \tag{1'}$$

The other modification is to use the reduced (direct) and diffuse intensity [14]. Defining

$$\left[\frac{1}{c}\frac{\partial}{\partial t}+\boldsymbol{\Omega}\cdot\nabla+\sigma_t\right]I_r(\boldsymbol{r},t,\boldsymbol{\Omega})=\varepsilon(\boldsymbol{r},t,\boldsymbol{\Omega}) \tag{23a}$$

and

$$I(\boldsymbol{r},t,\boldsymbol{\Omega})=I_d(\boldsymbol{r},t,\boldsymbol{\Omega})+I_r(\boldsymbol{r},t,\boldsymbol{\Omega}), \tag{23b}$$

the equation for $I_d(\boldsymbol{r},t,\boldsymbol{\Omega})$ becomes

$$\left[\frac{1}{c}\frac{\partial}{\partial t}+\boldsymbol{\Omega}\cdot\nabla+\sigma_t\right]I_d(\boldsymbol{r},t,\boldsymbol{\Omega}) - \frac{\sigma_s}{4\pi}\int I_d(\boldsymbol{r},t,\boldsymbol{\Omega}')p(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}')d\Omega' = \frac{\sigma_s}{4\pi}\int I_r(\boldsymbol{r},t,\boldsymbol{\Omega}')p(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}')d\Omega'. \tag{24}$$

I_r is a good approximation for early times; therefore, combining it with the late-time approximation has the potential of extending the range of validity of the approximation. Because the virtual source in Eq. (24) is a propagating source, Eq. (22) might give a good approximation when applied to Eq. (24). Considering the universality of the DE and the derivation approach used here, which is based on similar assumptions, the merit of the results should be judged relative to the DE. As a first check, we compare the result to the DE ($L=1$) using Eq. (14) [Eq. (A4) is exact in this case],

$$\left[1-\frac{1}{3}\frac{1}{\left(\frac{s_t}{c}+\sigma_t-\sigma_s p_1\right)\left(\frac{s_t}{c}+\sigma_t-\sigma_s p_0\right)}\nabla^2\right]\tilde{I}(\boldsymbol{r},s_t,\boldsymbol{\Omega}) = \tilde{\tilde{\Xi}}(\boldsymbol{r},s_t,\boldsymbol{\Omega}). \tag{25}$$

$\tilde{\tilde{\Xi}}(\boldsymbol{r},s_t,\boldsymbol{\Omega})$ is given by Eq. (A5). This is the Ishimaru [2] version of the DE. Equation (25) can be written as

$$\left[1-\frac{1}{3}\frac{c}{\sigma_s(p_0-p_1)}\left\{\frac{1}{\left(\frac{s_t}{c}+\sigma_t-\sigma_s p_1\right)} - \frac{1}{\left(\frac{s_t}{c}+\sigma_t-\sigma_s p_0\right)}\right\}\nabla^2\right]\tilde{I}(\boldsymbol{r},s_t,\boldsymbol{\Omega})=\tilde{\tilde{\Xi}}(\boldsymbol{r},s_t,\boldsymbol{\Omega}).$$

Ignoring the first term in curly brackets, we obtain the Furutsu approximation [5]. Using a higher-order small-k approximation, one generally does not obtain the telegraph equation. In the case where all the even and odd coefficients of the phase function are respectively equal, Eq. (A4) leads to the speed of propagation $v=c\sqrt{l/3}$. For $l>3$ the small-k approximation leads to a nonphysical early-time solution (see the discussion in Ref. [7]).

Equations (13) and (14) give the Lth-order approximation using a higher-order expansion in ∇^2. We expect the equation to be a good description for cases where the DE is almost a good description, i.e., late times, large distances, and a small-l–dominated phase function and source. The equation is expected to provide a better description than the DE of ''snake photons'' [15], which are early-arrived scattered photons used to improve image details. Such an equation, however, is mathematically hard to handle. Limiting cases of large and small k's that include only ∇^2 terms are desirable. A mathematical treatment of equations of this type is discussed in Appendix B: the equivalent wave field. The small-k approximation is developed in Appendix A. I have not developed a general large-k asymptotic expansion. Such an expansion can be obtained by keeping the two highest-order terms in ∇^2 while ignoring the lower-order terms. While the small-k approximation can be applied for late-time expansion whenever the Lth-order approximation is justified, the large-k expansion is justified only under limited circumstances. Such a case is the initial rise in photon transport in random media [1] at the point at which the diffusion approximation starts to break down.

Both Eqs. (15) and (22), which is an exact solution, and Eq. (A6), which is a late-time approximation, include an expansion of the source in a Laplace series, which is an expansion of the angular part of the Legendre polynomials. The appearance of Legendre polynomials in the solution is natural since the Legendre polynomials introduced in Eq. (3) are eigenvalues of the scattering phase function. The SHE was introduced in Eq. (4) only as a convenient intermediate step to describe the Legendre polynomials of the relative angle. Legendre polynomials of the relative angle were reintroduced in the final solution.

ACKNOWLEDGMENTS

This work was supported by Coastal Systems Station independent research. I would like to thank P. F. Schippnick for his help in the manuscript preparation and G. S. Sammelmann for a discussion of invariant properties of the determinant D.

APPENDIX A: THE SMALL-*k* APPROXIMATION

The high order of $\boldsymbol{s}$ in the polynomial makes the equation hard to deal with in most cases. A simpler approximation can be beneficial. Consistent with the diffusion approximation, we assume small spatial derivatives or small $\boldsymbol{s}$ (but not s_t). We will expand the determinant keeping zeroth- and second-order terms while keeping track of the fourth-order residual term. Using the first two terms of the expansion in s_z^2 in Eq. (13), which we call $\boldsymbol{d}_{m,L}$, we get

$$d_{m,L}=\left[1+s_z^2\sum_{l=|m|}^{L}\frac{l^2-m^2}{4l^2-1}\frac{1}{O_l O_{l-1}}\right]\prod_{l=|m|}^{L}O_l, \tag{A1}$$

where the fourth-order residual term is

$$s_z^4\left[\sum_{\substack{l,\lambda=|m|\\|l-\lambda|\leqslant 2}}^{L}\frac{l^2-m^2}{4l^2-1}\frac{\lambda^2-m^2}{4\lambda^2-1}\frac{1}{O_l O_{l-1}}\frac{1}{O_\lambda O_{\lambda-1}}\right]\times\prod_{l=|m|}^{L}O_l.$$

A weak requirement for the approximation to be valid is that $|s_z|\ll|O_l|$ or that

$$|s_z|\ll|s_t/c+\sigma_t(\boldsymbol{r})-\sigma_s(\boldsymbol{r})p_l(\boldsymbol{r})|, \tag{A2}$$

which applies for distances larger than the attenuation length or early times. The second-order approximation in s_z for $di_L=\det(\boldsymbol{D}_L)$ is obtained through a second-order expansion of the products $\Pi_{m=-L}^{L}d_{m,L}$,

$$di_L=\left[1-\frac{1}{3}\sum_{l=1}^{L}\frac{l}{O_lO_{l-1}}s_z^2\right]O_0\prod_{l=1}^{L}(O_l)^{2l}. \tag{A3}$$

The relation between the time and space derivatives is given by

$$d_L=1-\frac{1}{3}\sum_{l=1}^{L}\frac{l}{O_lO_{l-1}}s_z^2\equiv 1-\Psi(s_t)s_z^2. \tag{A4}$$

Equation (A4) is the small-gradient (k) approximation of Eq. (15).

We wish to expand Eq. (15) to second order in s_z. We use Eq. (18) as our starting point. We multiply both sides of Eq. (22) by a normalized second-order expansion of the determinant of $\boldsymbol{D}$. This is a somewhat arbitrary step, but it introduces the correct differential equation into the expansion. The adjugate matrix in Eq. (15) is one order lower in s_z than the determinant itself; therefore, we have to expand Eq. (21) to first order while neglecting the second order. The result in shorthand notation is

$$[1-\Psi(s_t)s_z^2]\tilde{I}=[\boldsymbol{O}^{-1}-\boldsymbol{O}^{-1}\cdot\boldsymbol{B}\cdot\boldsymbol{O}^{-1}]\cdot\tilde{\varepsilon}. \tag{A5}$$

Using Eq. (22) and expanding $\boldsymbol{O}^{-1}\boldsymbol{B}\boldsymbol{O}^{-1}$ into spherical harmonics and collecting terms, we can reformulate Eq. (18) as

$$[1-\Psi(\boldsymbol{r},s_t)\nabla^2]\tilde{I}(\boldsymbol{r},s_t,\boldsymbol{\Omega})=\tilde{\Xi}_1(\boldsymbol{r},s_t,\boldsymbol{\Omega})+(\boldsymbol{\Omega}\cdot\nabla)\tilde{\Xi}_2(\boldsymbol{r},s_t,\boldsymbol{\Omega}), \tag{A6}$$

$$\Psi(\boldsymbol{r},s_t)=\frac{1}{3}\sum_{l=1}^{L}\frac{l}{\left[\frac{s_t}{c}+\sigma_t(\boldsymbol{r})-\sigma_s(\boldsymbol{r})p_l(\boldsymbol{r})\right]\left[\frac{s_t}{c}+\sigma_t(\boldsymbol{r})-\sigma_s(\boldsymbol{r})p_{l-1}(\boldsymbol{r})\right]}, \tag{A6a}$$

$$\tilde{\Xi}_1(\boldsymbol{r},s_t,\boldsymbol{\Omega})=\sum_{l=0}^{L}\frac{\tilde{\varepsilon}_l(\boldsymbol{r},s_t,\boldsymbol{\Omega})}{\left[\frac{s_t}{c}+\sigma_t(\boldsymbol{r})-\sigma_s(\boldsymbol{r})\cdot p_l(\boldsymbol{r})\right]}, \tag{A6b}$$

$$\tilde{\Xi}_2(\boldsymbol{r},s_t,\boldsymbol{\Omega})=\sum_{l=0}^{L}\frac{\tilde{\varepsilon}_l(\boldsymbol{r},s_t,\boldsymbol{\Omega})}{\left[\frac{s_t}{c}+\sigma_t(\boldsymbol{r})-\sigma_s(\boldsymbol{r})p_l(\boldsymbol{r})\right]\left[\frac{s_t}{c}+\sigma_t(\boldsymbol{r})-\sigma_s(\boldsymbol{r})p_{l-1}(\boldsymbol{r})\right]}, \tag{A6c}$$

$$\tilde{\varepsilon}_l(\boldsymbol{r},s_t,\boldsymbol{\Omega})=\int\tilde{\varepsilon}(\boldsymbol{r},s_t,\boldsymbol{\Omega})\;\;P_l(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}')d\Omega'. \tag{A6d}$$

Equation (A6) is similar to the DE in that it includes only second-order derivatives in $\boldsymbol{r}$; however, it accounts for higher-order terms of the expansion of the phase function and applies to the intensity I rather than to the density. Its limitation is that it uses the small-gradient approximation, which limits its validity to later times. An interesting observation here is that when all the even and odd components of p_l are equal we can use the solution given in Ref. [7], which leads to propagation of the front with speed $\sqrt{(2l+1)/3}c$, where l is the order of the expansion. The fact that the apparent speed of propagation is faster than the speed of light is an artifact of the small-k approximation, which is invalid at early times.

APPENDIX B: THE EQUIVALENT WAVE FIELD

Equation (A6) in the large-k asymptotic expansion has the form

$$[1-\Psi(s_t)\nabla^2]\tilde{I}(\boldsymbol{r},s_t,\boldsymbol{\Omega})=\tilde{\Xi}(\boldsymbol{r},s_t,\boldsymbol{\Omega}), \tag{B1}$$

which, as can be shown using arguments similar to those used above, is quite common as a second-order approximation in many physical phenomena. We will show that the solution $I(\boldsymbol{r},t,\boldsymbol{\Omega})$ can be related to the solution of an equivalent wave equation

$$\left[1-\frac{1}{p_t^2}\nabla^2\right]\tilde{I}_w(\boldsymbol{r},p_t,\boldsymbol{\Omega})=\tilde{\Xi}_w(\boldsymbol{r},p_t,\boldsymbol{\Omega}). \tag{B2}$$

Such a relation is very beneficial due to the mathematical properties of the wave equation that make it useful in remote sensing and also relatively easy to solve in both direct and inverse problems. For the sake of simplicity, we ignore some of the mathematical subtleties. For compactness, we omit the arguments $\boldsymbol{r}$ and $\boldsymbol{\Omega}$:

$$\Psi(g(p_t))=\frac{1}{p_t^2}, \tag{B3}$$

$$\tilde{I}(\sqrt{g(p_t)})=\tilde{I}_w(p_t),\quad \tilde{\Xi}(\sqrt{g(p_t)})=\tilde{\Xi}_w(p_t). \tag{B4}$$

Using the forward and inverse Laplace transform

$$\tilde{I}_w(p_t)=\int_0^\infty I(t)\exp[-\sqrt{g(p_t)}t]dt, \tag{B5}$$

$$I_w(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\tilde{I}_w(p_t)\exp(p_t t)dp_t, \tag{B6}$$

and performing the two integrations after reversing the order of integration, we obtain

$$I_w(t)=\int_0^\infty I(\tau)K(t,\tau)d\tau, \quad \Xi_w(t)=\int_0^\infty \Xi(\tau)K(t,\tau)d\tau, \tag{B7}$$

$$K(t,\tau)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \exp[-\sqrt{g(p_t)}\tau]\exp(p_t t)dp_t. \tag{B8}$$

If the boundary conditions are frequency dependent, this fact must be accounted for. Equations (B7) and (B8) usually represent an ill-posed problem and one must exercise caution when using them. However, with proper care it is possible to obtain an optimal form of the kernel $K(t,\tau)$ for practical use. An inverse relation of the form

$$I(t)=\int_0^\infty I_w(t)K(t,\tau)d\tau \tag{B9}$$

can also be derived. (For a discussion of the equivalent wave field in the DE limit, see Ref. [16].)

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